

In the Name of GOD

As you know, in digital Comm. Systems, we use Digital Modulation to send our digital data stream, on electromagnetic waveforms on the Comm. channel.

Digital modulation is an interface between digital blocks of comm. systems and physical channel. So, digital modulation prepare a signal (waveform) compatible with channel characteristics and channel BW, to transfer data from transmitter to receiver.

In digital modulation techniques, we use some important parameters, related to the signal space and vector space of modulated signals. So, we first introduce these parameters.

## 1) Signal Energy

for the set of signals  $\{S_m(t)\}_{m=1}^M$ , signal energies are defined as below real signal

$$E_m = \underbrace{\int_{-\infty}^{\infty} |S_m(t)|^2 dt}_{\text{the energy of } S_m(t) \text{ (band-pass signal)}} = \int_{-\infty}^{\infty} S_m(t)^2 dt = \frac{1}{2} \int_{-\infty}^{\infty} |S_p(t)|^2 dt$$

$\hookrightarrow$  the energy of  $S_m(t)$  (band-pass signal)

equivalent  
low-pass signal

$$E_m = \int_{-\infty}^{\infty} |S_m(t)|^2 dt = \frac{1}{2} \int_{-\infty}^{\infty} |S_{l_m}(t)|^2 dt$$

$$E_m = \left\| \underline{S}_m \right\|^2 = \sum_{i=1}^n |S_{m_i}|^2 = \langle \underline{S}_m, \underline{S}_m \rangle$$

↓  
Vector rep. of  $S_m(t)$  in equ. vector space

$$\underline{S}_m = \begin{pmatrix} S_{m_1} & S_{m_2} & \dots & S_{m_n} \end{pmatrix}^T$$

## 2) Normalized Correlation function

For any two signal  $s_m(t)$  and  $s_n(t)$ , normalized correlation function is defined as follows,

$$\text{Normalized corr. func.} = \frac{1}{\sqrt{E_m E_n}} \int_{-\infty}^{\infty} s_m(t) \overbrace{s_n^*(t)}^{S_n(t)} dt$$

band-pass signals are real signals

$$= \operatorname{Re} \left\{ \frac{1}{2\sqrt{E_m E_n}} \int_{-\infty}^{\infty} \underbrace{s_{p_m}(t)}_{\text{low-pass}} \underbrace{s_{p_n}^*(t)}_{\text{low-pass}} dt \right\}$$

equivalent low-pass signals

$$\text{Normalized Corr. function} = \frac{1}{\sqrt{E_m E_n}} \int_{-\infty}^{\infty} S_m(t) S_n(t) dt$$

$$= \operatorname{Re} \{ \rho_{mn} \}$$

$$= \frac{\langle \underline{S}_m, \underline{S}_n \rangle}{\| \underline{S}_m \| \| \underline{S}_n \|} = C \langle \underline{S}_m, \underline{S}_n \rangle$$

↑  
in equ. vector Space

$$\Rightarrow -1 \leq \operatorname{Re} \{ \rho_{mn} \} \leq 1 \quad \left( \begin{array}{l} \text{This is a simple rep. of Schwartz} \\ \text{inequality} \end{array} \right)$$

### 3) Euclidean Distance between two signals

For any two signals in the signal space, the Euclidean Distance is defined as below;

$$(e) \quad d_{mn} = \left[ \int_{-\infty}^{\infty} (s_m(t) - s_n(t))^2 dt \right]^{\frac{1}{2}}$$

$$\stackrel{\uparrow}{=} \| \underline{s}_m - \underline{s}_n \| = \left( \langle \underline{s}_m - \underline{s}_n, \underline{s}_m - \underline{s}_n \rangle \right)^{\frac{1}{2}}$$

Vector Space

$$\Rightarrow d_{mn}^{(e)} = \left\langle \underline{S}_m - \underline{S}_n, \underline{S}_m - \underline{S}_n \right\rangle^{\frac{1}{2}}$$

$$= \left( \underbrace{\left\langle \underline{S}_m, \underline{S}_m \right\rangle + \left\langle \underline{S}_n, \underline{S}_n \right\rangle}_{\uparrow} - 2 \left\langle \underline{S}_m, \underline{S}_n \right\rangle \right)^{\frac{1}{2}}$$

$\underline{S}_m$  and  $\underline{S}_n$  are  
real vectors related  
to real signals  
 $s_m(t)$  and  $s_n(t)$

$$\Rightarrow d_{mn}^{(e)} = \left( \|\underline{S}_m\|^2 + \|\underline{S}_n\|^2 - 2 \underbrace{\left\langle \underline{S}_m, \underline{S}_n \right\rangle}_{\text{Re}\{P_{mn}\}} \right)^{\frac{1}{2}} \| \underline{S}_m \| \| \underline{S}_n \|$$

$$\Rightarrow d_{mn}^{(e)} = \left( \|\underline{s}_m\|^2 + \|\underline{s}_n\|^2 - 2 \|\underline{s}_m\| \|\underline{s}_n\| \operatorname{Re}\{\rho_{mn}\} \right)^{1/2}$$

$$\Rightarrow d_{mn}^{(e)} = \left( E_m + E_n - 2 \sqrt{E_m E_n} \operatorname{Re}\{\rho_{mn}\} \right)^{1/2}$$

So, we can conclude that the Euclidean Distance between any two signals in the signal space is related to these signals Energies and their Correlation function.

If signals  $s_m(t)$  and  $s_n(t)$  have equal energies,  
means ,  $E_m = E_n = E$  then we have,

$$d_{mn}^{(e)} = \left( E_m + E_n - 2 \sqrt{E_m E_n} \operatorname{Re} \{ \rho_{mn} \} \right)^{\frac{1}{2}}$$

$$= \left( E + E - 2 \underbrace{\sqrt{E^2}}_E \operatorname{Re} \{ \rho_{mn} \} \right)^{\frac{1}{2}} = \left( 2E - 2E \operatorname{Re} \{ \rho_{mn} \} \right)^{\frac{1}{2}}$$

$$E_m = E_n = E$$

$$d_{mn}^{(e)} = \sqrt{2E} \left( 1 - \operatorname{Re} \{ \rho_{mn} \} \right)^{\frac{1}{2}}$$

for signals with  
equal energies

In the case of equal energy signals, when the  $d_{mn}^{(e)}$  is maximized?

We know that

$$d_{mn}^{(e)} = \sqrt{2\epsilon} \left( 1 - \operatorname{Re} \{ P_{mn} \} \right)^{\frac{1}{2}} \quad \textcircled{1}$$

on the other hand, we know that

$$-1 \leq \operatorname{Re} \{ P_{mn} \} \leq 1 \quad \textcircled{2}$$

So,  $d_{mn}^{(e)}$  is maximized when  $\operatorname{Re} \{ P_{mn} \} = -1$

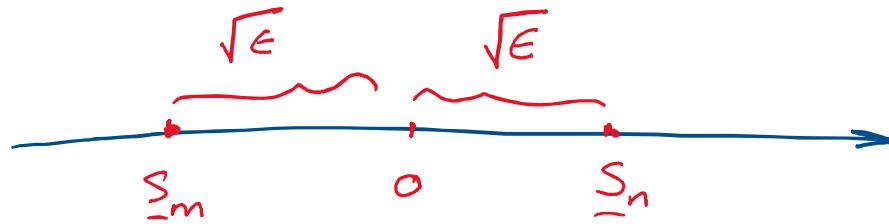
$$\Rightarrow d_{mn}^{(e)} \Big|_{\max} = \sqrt{2E} (1+1)^{\frac{1}{2}} = 2\sqrt{E}$$

This kind of signals is named as antipodal signals

Means, for antipodal signals,  $s_m(t)$  and  $s_n(t)$ ,

they have equal energies and they are in the same line with different directions or equivalently  $s_m(t) = -s_n(t)$

antipodal  
signals



$$\begin{aligned}\underline{s}_m &= -\underline{s}_n \\ d_{mn}^{(e)} &= 2\sqrt{E}\end{aligned}$$

As you may know, the Euclidean distance between signals in the signal space of Digitally modulated Signal is a factor that affect the performance of modulated signals.

The more distance between signal, the better Performance of modulation technique.

So, to better evaluation of the Modulation Performance we define the parameter  $d_{min}$  as the Minimum Euclidean distance between signal  $\{S_m(t)\}_{m=1}^M$ , as below

$$d_{\min} = \text{Min} \left\{ d_{mn}^{(e)} \mid \forall s_m(t), s_n(t) \in \left\{ s_m(t) \right\}_{m=1}^M, m \neq n \right\}$$

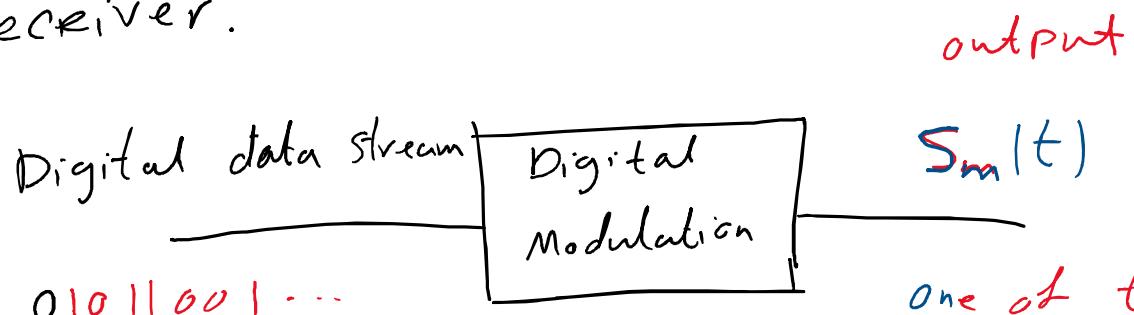
for any signal set  $\left\{ s_m(t) \right\}_{m=1}^M$

Now, we want to talk about digital Modulation.

In any Digital Modulation technique, we want to transmit our digital data stream, from a transmitter to receiver using the proper waveforms or signals.  
 (we know that these signals are band-pass signals)

Means that we want to modulate some band-pass signals using our digital data stream.

So, in M ary digital modulation technique, we have a set of M band-pass signals as  $\{S_m(t)\}_{m=1}^M$  to carry our information from transmitter to the receiver.



one of the signals in the set  $\{S_m(t)\}_{m=1}^M$

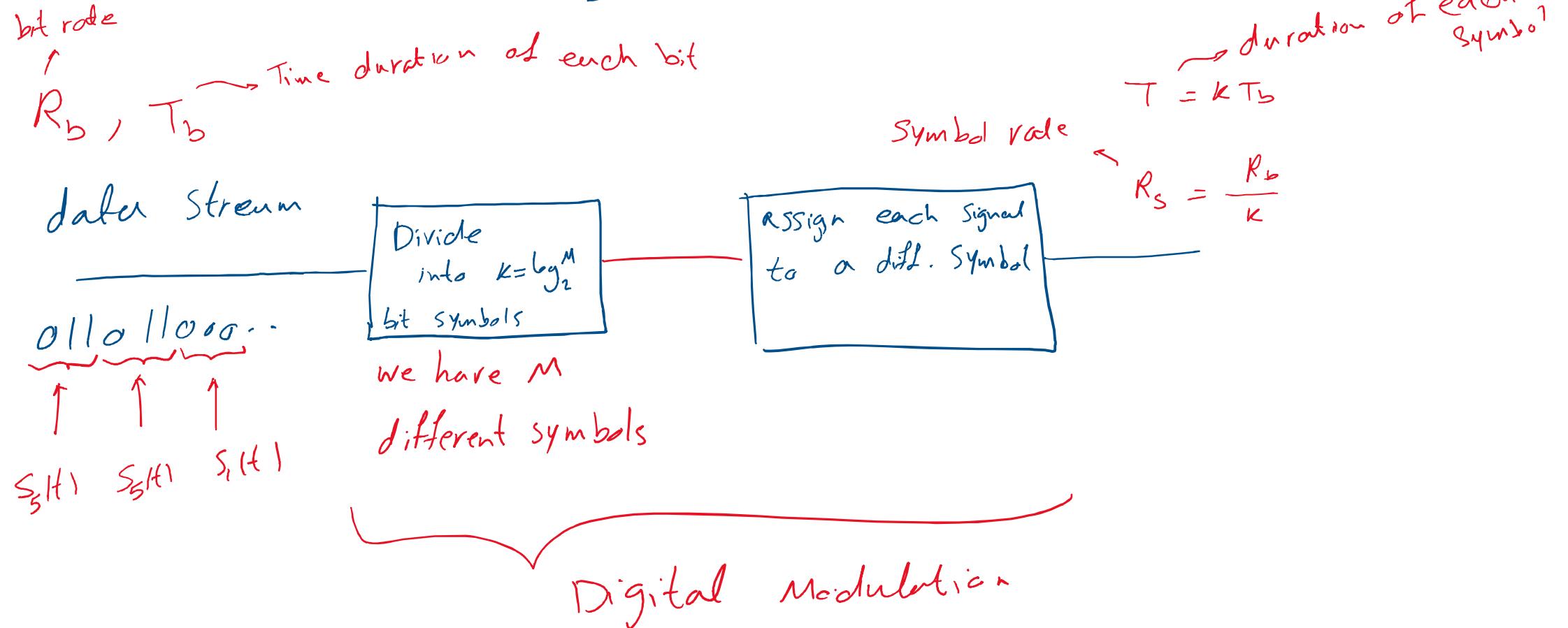
So, we have  $M$  different signals  $s_1(t), s_2(t), \dots, s_M(t)$   
 $\left( \{s_m(t)\}_{m=1}^M \right)$  to carry our information. Therefore we

can carry  $M$  different symbols, using these signals.

So, we can transmit  $M$  different symbols, each symbol has the length equal to  $K = \log_2^M$  bits.

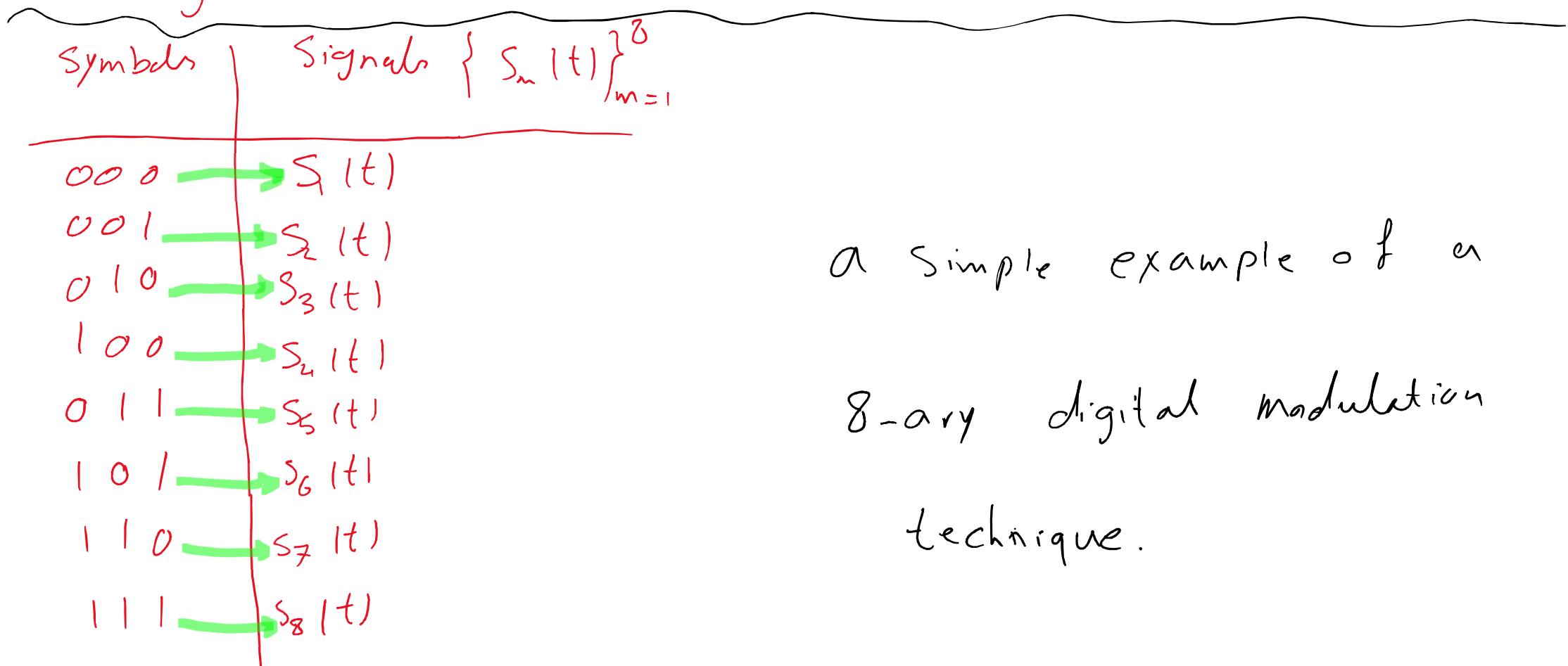
So, to modulate our data stream, first we should divide our stream into  $k$ -bit symbols and then assign a signal

of set  $\{S_m(t)\}_{m=1}^M$  to any symbols.



for example if  $M=8$  then we should divide or data stream into

$$K = \log_2 8 = 3 - \text{bit symbols}$$



We have 8 symbol

If our signal assignment to each symbol at time  $t$ , is just related to the symbol in time  $t$ , we call the Modulation technique as Memory less. otherwise if the signal assignment to each symbol at time  $t$ , related to this symbol and also  $(L-1)$  previous symbols, we have a not memory-less modulation.

If our signal assignment to symbols follow the superposition rule, we call the Modulation technique as linear. otherwise the modulation is called non-linear.

In this course we focus on linear Memory-less digital modulation techniques.

In digital M-ary Modulation we need  $M$  different band pass signals  $\{S_m(t)\}_{m=1}^M$  to carry our information.

It is difficult to find  $M$  different signals, with proper characteristics, especially when  $M$  is a large number.

So, what we should do?

We select some limited number of orthonormal signal set and (complete)

use the parameters such as amplitude, phase, frequency or a combination of these parameters to make M different signals  $\{S_m(t)\}_{m=1}^M$ .

Usually in linear memory-less modulation we use the set of complete orthonormal functions as below

$$f_1(t) = \sqrt{\frac{2}{T}} \cos 2\pi f_c t, \quad f_2(t) = -\sqrt{\frac{2}{T}} \sin 2\pi f_c t$$

in which T is the symbol duration in time.

As you know the  $\{f_i(t)\}_{i=1}^{\infty}$  make a complete orthonormal basis functions in signal set. So, any signal in the signal space can rep. as a linear combination of these two signals.

Besides; from the rep. of any band-pass signal  $s(t)$  based on its equivalent low-pass signal  $s_e(t)$  we know that;

$$s(t) = \operatorname{Re} \left\{ s_e(t) e^{j2\pi f_c t} \right\}$$

$$s(t) = x(t) \cos 2\pi f_c t - y(t) \sin 2\pi f_c t$$

$$s(t) = a(t) \cos(2\pi f_c t + \theta(t))$$

$$\left\{ \begin{array}{l} s_e(t) = x(t) + j y(t) \\ a(t) = (x^2(t) + y^2(t))^{1/2} \\ \theta(t) = \tan^{-1} \frac{y(t)}{x(t)} \end{array} \right.$$

So, any signal in the signal set  $\{s_m(t)\}_{m=1}^{\infty}$  can be represented by the use of  $f_1(t)$  and  $f_2(t)$

$$f_1(t) = \sqrt{\frac{2}{T}} \cos 2\pi f_c t , \quad f_2(t) = -\sqrt{\frac{2}{T}} \sin 2\pi f_c t$$

$$\Rightarrow s_m(t) = \underbrace{x(t)}_{(\text{I}) \text{ Inphase}} f_1(t) + \underbrace{y(t)}_{\text{Quadrature (Q)}} f_2(t) : \text{Modulated Signal}$$

or equivalently

$$s_{e_m}(t) = x(t) + j y(t)$$

So, we call  $f_1(t)$  and  $f_2(t)$  as Carrier signals or Carrier waveforms.

Means our information on  $\overset{\text{I}}{\sim}x(t)$  and  $\overset{\text{Q}}{\sim}y(t)$  (Quadrature part of transmitted signals) are modulated using Carrier waveforms,  $f_1(t)$  and  $f_2(t)$  and make modulated signal  $s_m(t)$  to transmit on the Comm. channel

If the information rate at the input of digital modulation is equal to  $R$  bits/s then the symbol rate of modulated signal will be equal

to 
$$R_s = \frac{R}{k} = \frac{R}{\log_2^m}$$

We have  $k$ -bit symbols

If the time duration of each symbol is equal to  $T$ , then the <sup>time</sup>duration of each bit will be equal to

$$T = T_b(k)$$

bit duration time

Symbol  
duration  
time

$$\Rightarrow \left\{ \begin{array}{l} T_b = \frac{T}{k} = \frac{T}{6g_2^M} \\ R_s = \frac{R_b}{k} = \frac{R_b}{6g_2^M} \end{array} \right. \Rightarrow \underbrace{R_b}_{\text{bit rate}} = k R_s$$

Symbol Rate